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Gross(2001) Let B be a tropical affine manifold(a manifold with transitional maps in $\mathbb{R}^n \rtimes \mathrm{GL}_n(\mathbb{Z}) \subset \mathrm{Aff}(\mathbb{R}^n)$), $\exists$ local system $\Lambda \subset \mathcal{T}_B$ locally generated by $\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, ...\frac{\partial}{\partial y_n}, \{y_i\}$ are local coordinates. We also have $\check{\Lambda} \subset \mathcal{T}_B^*$ generated by $dy_1, ...dy_n$. These Λ and $\check{\Lambda}$ are well defined.

Define $X(B) := \mathcal{T}_B/\Lambda$ and $\check{X}(B) = \mathcal{T}_B^*/\check{\Lambda}$ over B.Then $X(B)$ and $\check{X}(B)$ are complex manifold and symplectic manifold respectively. This is semi-flat SYZ. A toy picture. Not possible for more complex examples because they all have Euler characteristics 0.

To allow singular fibers, let $B_0 \subset B$ where B_0 has an affine structure, $\Delta := B\backslash B_0$ codim $= 2$. So after compactifying the spaces we have:

$$\begin{array}{ccccccc} X(B_0) & \subset & X(B) & & \check{X} & \subset & \check{X}(B_0) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ B_0 & \subset & B & & B_0 & \subset & B, \end{array}$$

wishing that $X(B)$ is a complex manifold and $\check{X}(B)$ is a symplectic manifold.

- symplectic world: is true. Castano-Bernard-Matessi dim $= 3$.
- complex world: known that this doesn't happen.

Now we modify $\epsilon > 0$, $X_\epsilon(B_0) = \mathcal{T}_{B_0}/\epsilon\Lambda$. When $\epsilon \to 0$:, this is the large complex structure limit.

Goal: modify complex structure for small ϵ(Fukaya 2001, Chan,Leung,Ma). Seems very hard.

Local model for degeneration($\epsilon \to 0$) is $\mathbb{C}^{n+1} \to \mathbb{C}, (x_0, x_1, ..., x_n) \to x_0x_1...x_n$.

Exercise. *Take* $0 < \delta < 1, t \in \mathbb{C}^*$. *Let* $N_{\delta,t} := \{(x_0, ..., x_n) \in \mathbb{C}^{n+1} ||x| < \delta, \prod x_i = t\}$. *We have* T^n *fibration* $N_{\delta,t} \to \mathbb{R}^n, (x_0, ..., x_n) \to (\frac{log|x_1|}{log|t|}, ..., \frac{log|x_n|}{log|t|})$. *There is a large open subset* U *of standard simplex*

$$Conv\{0, (1, ..., 0), ..., (0, ..., 1)\}$$

such that $N_{\delta,t} \cong X_\epsilon(U), \epsilon^{-1} = -\frac{log|t|}{2\pi}$.

Exercise. *Generalize the statement to monomial morphism* $X_\sigma \to \mathbb{C}$ *given by* m. *Analog of* $N_{\delta,t} \cong X_\epsilon(U)$, U *is a large open subset of* $\sigma \cap \{\langle m, \cdot\rangle = 1\}$.

Moral:(First discussions with Bernd) Understand B by considering toric degenerations $\mathcal{X} \to \mathcal{C}$ which locally looks toric.

Bernd(2000, work with Schröer): observed interchange under mirror symmetry of data controlling irreducible components of a degeneration and data controlling 0-dimentional strata of mirror. Logarithmic geometry.

Log geometry

Definition. *A log structure on a scheme* X *is data of*

- $\mathcal{M}_X$ *a sheaf of (commutative, with unit) monoids on* X

- *$\alpha_X : \mathcal{M}_X \to \mathcal{O}_X$ a monoid homomorphism with monoid structure in $\mathcal{O}_X$ given by multiplication such that $\alpha_X : \alpha_X^{-1}(\mathcal{O}_X^*) \to \mathcal{O}_X^*$ is an isomorphism.*

We call the triple $(X, \mathcal{M}_X, \alpha_X)$ a log scheme. A morphism of log schemes $f : (X, \mathcal{M}_X, \alpha_X) \to (Y, \mathcal{M}_Y, \alpha_Y)$ is a scheme morphism $f : X \to Y$ along with $f^\flat : f^{-1}\mathcal{M}_Y \to \mathcal{M}_X$ with

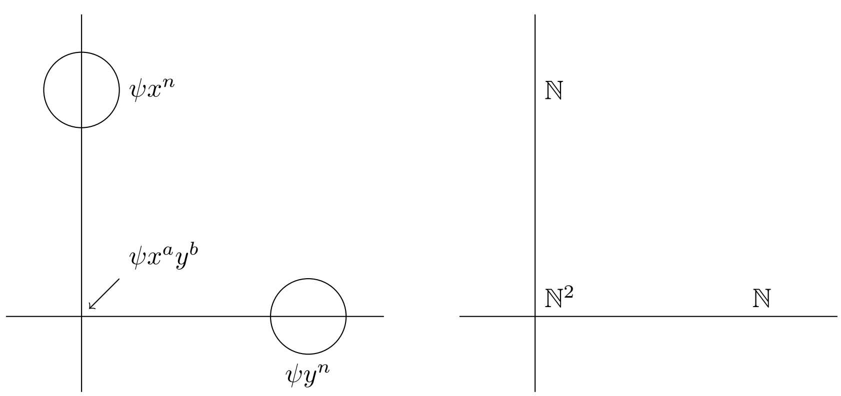
$$\begin{array}{ccc} f^{-1}\mathcal{M}_Y & \xrightarrow{f^\flat} & \mathcal{M}_X \\ {\scriptstyle \alpha_Y}\downarrow & & \downarrow{\scriptstyle \alpha_X} \\ f^{-1}\mathcal{O}_Y & \xrightarrow{f^*} & \mathcal{O}_X. \end{array}$$

Key examples:

- The divisorial log structure. Let $D \subset X$ be a divisor, $j : X \backslash D \hookrightarrow X$, $\mathcal{M}_X := (j_* \mathcal{O}_{X \backslash D}^*) \cap \mathcal{O}_X$ which is the sheaf of regular functions on X invertible on $X \backslash D$. $\alpha_X : \mathcal{M}_X \hookrightarrow \mathcal{O}_X$ is the obvious inclusion. Note if given pairs (X, D) and (Y, E) the $f : X \to Y$ s.t. $f^*\phi$ vanishes only on D if ϕ vanishes only on E, i.e. $f^{-1}(E) \subset D$ then $f : X \to Y$ is a log morphism.

- $\mathrm{Spec} k^\dagger$ Standard log point. $X = \mathrm{Spec} k$, $\mathcal{M}_{\mathrm{Spec} k} := k^* \oplus \mathbb{N}$ and
$$\alpha(r, n) = \begin{cases} r, & \text{if } n = 0 \\ 0, & \text{if } n \neq 0 \end{cases}$$

- Pullback log structure. $f : X \to Y$ a scheme morphism, $\mathcal{M}_Y, \alpha_Y$ a log structure on Y. Define a log structure on X,
$$f^*\mathcal{M}_Y := (f^{-1}\mathcal{M}_Y \oplus \mathcal{O}_X^*)/\sim$$
with $(p, 1) \sim (1, f^*(\alpha_Y(p)))$ if $\alpha_Y(p) \in \mathcal{O}_Y^*$. $f^{-1}\mathcal{M}_Y \xrightarrow{\alpha_Y} f^{-1}\mathcal{O}_Y \xrightarrow{f^*} \mathcal{O}_X$.
How to visualize log structure? Ghost sheaf: Given $(Y, \mathcal{M}_Y)$, let $\overline{\mathcal{M}}_Y = \mathcal{M}_Y/\alpha^{-1}(\mathcal{O}_Y^*)$. $1 \to \mathcal{O}_Y^* \xrightarrow{\alpha^{-1}} \mathcal{M}_Y \to \overline{\mathcal{M}}_Y \to 0$.
E.g. $Y = (\mathbb{A}^2, V(xy))$.



$\overline{\mathcal{M}}_Y = \mathbb{N}_{V(x)} \oplus \mathbb{N}_{V(y)}$.

Fact: $\overline{f^*\mathcal{M}_Y} \cong f^{-1}\overline{\mathcal{M}}_Y$.

e.g. $X = V(xy)$ with pull back log structure. Then $\overline{\mathcal{M}}_X$ is scheme which knows it is sitting inside $\mathbb{A}^2$.

- Let P be a toric monoid, i.e. a rational polyhedral cone $\sigma \subset M_{\mathbb{R}}$ and $P = \sigma \cap M$. $X = \mathrm{Spec}k[P] \supset U$ big torus orbit. $D = X\backslash U$ =union of toric divisors on X. A log scheme is said to be fine saturated(fs) if (étale) locally it arises as a pull back via a morphism $X \to \mathrm{Spec}k[P]$. Note specifying such morphism is the same as giving a map $P \to \mathcal{O}_X \rightsquigarrow k[P] \to \mathcal{O}_X \rightsquigarrow X \to \mathrm{Spec}k[P]$. Pullback toric log structure. The map $P \to \mathcal{O}_X$ is called a chart for the log structure.

Why Log structure? Kato: Log structures are "magic powder" making singular varieties smooth. We can translate all properties of schemes into properties of log schemes. And we have a notion of log smooth.

Étale locally the log smooth morphisms look like

$$\begin{array}{ccc} X & \longrightarrow & \mathrm{Spec}k[P] \\ \downarrow f & & \downarrow f' \\ Y & \longrightarrow & \mathrm{Spec}k[Q] \end{array}$$

with f' induced by $\theta : Q \to P$ injective. $k[Q] \to k[P]$ such that induced morphism $X \to Y \times_{\mathrm{Spec}k[Q]} \mathrm{Spec}k[P]$ is smooth in the ordinary sense.

Example.

$$\begin{array}{ccc} X = \mathbb{A}^{n+1} & \xrightarrow{=} & Speck[\mathbb{N}^{n+1}] \\ \downarrow & & \downarrow \\ Y = \mathbb{A}^1 & \xrightarrow{=} & Speck[\mathbb{N}] \end{array}$$

induced by $\mathbb{N} \to \mathbb{N}^{n+1}, 1 \to (1, ..., 1)$. *Now this is log smooth. Pullback this to*

$$\begin{array}{ccc} V(x_0...x_n) & \longrightarrow & \mathbb{A}^{n+1} \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathbb{A}^1. \end{array}$$

Then $V(x_0, ..., x_n)$ *is also log smooth.*

Exercise. *Check the log structure on* 0 *is the standard log structure* $Speck^\dagger$.

Common situation:

$$\begin{array}{ccc} \mathcal{X} & \supseteq & \mathcal{X}_0 \\ \downarrow & & \downarrow \\ \mathcal{C} & \supseteq & 0 \end{array}$$

degeneration of Calabi-Yau. Give $(\mathcal{X}, \mathcal{X}_0) \to (\mathcal{C}, 0)$ divisorial log structure. Pullback of the log structure to $\mathcal{X} \to 0$ is still viewed as smooth!

E.g.$\mathcal{X} = V(tf_4 + x_0x_1x_2x_3) \subset \mathbb{P}^3 \times \mathbb{A}^1$, degeneration of $K3$ surfaces. $\mathcal{X}_0 = \mathbb{P}^2 \cup \mathbb{P}^2 \cup \mathbb{P}^2 \cup \mathbb{P}^2$. $\mathcal{X}_0 \leftrightarrow \check{\mathcal{X}}_0$ central fiber of a mirror family. Exchanges combinatorial information about components and log structure at 0-dimensional strata $\mathcal{X}$ has 24 singularities along $\mathrm{Sing}(\mathcal{X}_0)$, so the log structure is not fs at these points.

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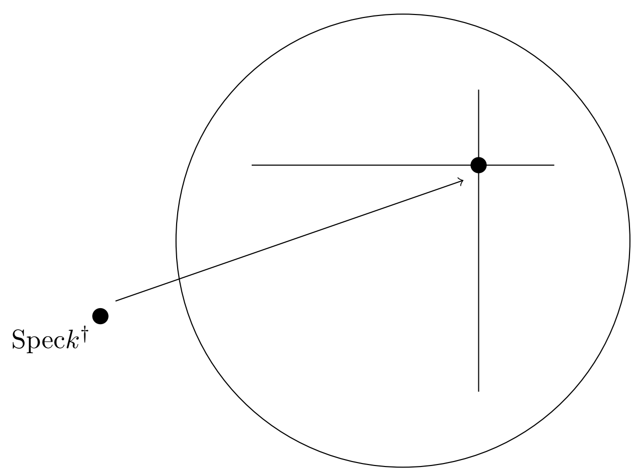
Continue the introduction to log geometry.

- $X \xrightarrow{f} \mathrm{Spec} k^\dagger$, standard log point. $f^\flat : f^{-1}(k^* \oplus \mathbb{N}) \to \mathcal{M}_X$. $f^{-1}(k^* \oplus 0)$ maps to $\alpha_X^{-1}(k*) \subseteq \mathcal{O}_X^*$. Only need to know what $\rho = f^\flat(1,1)$ is, must satisfy $\alpha_X(\rho) = 0$. This is the only information.

- $\mathrm{Spec} k^\dagger \xrightarrow{f} X$ gives a point $x \in X$. $f^\flat : f^{-1}\mathcal{M}_X = \mathcal{M}_{X,x} \to k^* \oplus \mathbb{N}$. $\mathcal{M}_{X,x} = \mathcal{O}_{X,x}^* \oplus \overline{\mathcal{M}}_{X,x}$. Determined by a map $\overline{\mathcal{M}}_{X,x} \to k^* \oplus \mathbb{N}$, i.e. an element of $\mathrm{Int}(\mathrm{Hom}(\overline{\mathcal{M}}_{X,x}, \mathbb{N}))$ and an element of the algebraic torus $\mathrm{Hom}(\overline{\mathcal{M}}_{X,x}, k^*)$. Let $X = (\mathbb{A}^2, V(xy))$.



$\overline{\mathcal{M}}_{\mathbb{A}^2,0} = \mathbb{N}^2 \to \mathcal{M}_{\mathbb{A}^2,0}, (\alpha, \beta) \mapsto x^\alpha y^\beta$. The map $\mathbb{N}^2 \to \mathbb{N}$ is determined by $(a, b) \in \mathbb{N}^2$ by $(\alpha, \beta) \mapsto a\alpha + b\beta$, neither a nor b can be 0.

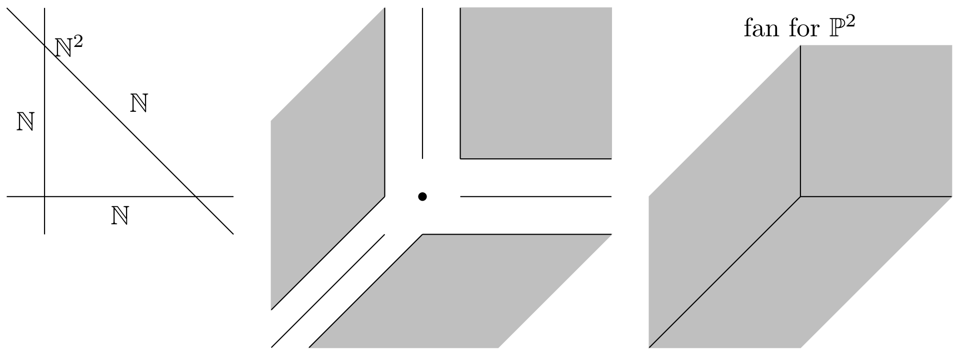
Now we are going to talk about tropicalization and see what the extra information in log geometry mean.

Tropicalization of (fs) log schemes

Let X be a fs log scheme, $x \in X$, $\sigma_X = \mathrm{Hom}(\overline{\mathcal{M}}_{X,x}, \mathbb{R}^+_{\geq 0})$. If $x, y \in X$, $x \in \{y\}^-$, then there is a generalization map

$$\overline{\mathcal{M}}_{X,x} \to \overline{\mathcal{M}}_{X,y}$$

which induces $\sigma_y \to \sigma_x$ which is an inclusion of faces(fs condition). Define $\Sigma(X)$ to be the polyhedral cone complex obtained by gluing all σ_x via these face maps. e.g. $(\mathbb{P}^2, L_0 \cup L_1 \cup L_2)$.

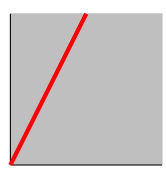


Exercise. *If one applies this construction to a tori variety X with standard toric log structure, then you get the fan for X, as abstract polyhedral complex.*

Functorial: $f : X \to Y$, $\overline{f}^\flat : \overline{\mathcal{M}}_{Y,f(x)} \to \overline{\mathcal{M}}_{X,x}$. $\Sigma(f) : \sigma_x \to \sigma_y$. Glue together we get $\Sigma(f) : \Sigma(X) \to \Sigma(Y)$. E.g. $\mathrm{Spec} k^\dagger \to \mathbb{A}^2$.

$$\begin{array}{ccc} \mathrm{Hom}(\mathbb{N}, \mathbb{R}_{\geq 0}) = \mathbb{R}_{\geq 0} & \to & \mathrm{Hom}(\mathbb{N}^2, \mathbb{R}_{\geq 0}) = \mathbb{R}^2_{\geq 0} \\ 1 & \mapsto & (a,b) \end{array}$$

where (a,b) determines the map $\overline{\mathcal{M}}_{\mathbb{A}^2,0} \xrightarrow{(a,b)} \mathbb{N}$.



$\Sigma(\mathbb{A}^2)$

Subdividing $\mathbb{R}^2_{\geq 0}$ gives a toric blowup of $\mathbb{A}^2$. An element of $\mathrm{Hom}(\overline{\mathcal{M}}_{\mathbb{A}^2,0}, k^*) = (k^*)^2$ determines a point on exceptional divisor. This describes all morphisms $\mathrm{Spec} k^\dagger \to \mathbb{A}^2$ up to isomorphisms of $\mathrm{Spec} k^\dagger$. There is k^* worth of automorphism.

$$\begin{array}{ccc} \mathrm{Spec} k^\dagger & \longrightarrow & \mathbb{A}^2 \\ \downarrow \cong & \nearrow & \\ \mathrm{Spec} k^\dagger & & \end{array}$$

<u>Log smooth curves</u>

Let $\pi : C \to W$ be a log morphism [convention: write $\underline{C}, \underline{W}$ for the underlying schemes] such that

- π is log smooth and flat

- All scheme theoretical fibers are reduced and dimension 1

We call this a family of log curves. Description of log curves over $\underline{W} = \mathrm{Spec} A$. A a complete local ring, with log structure coming from a chart $\varphi : Q \to A$ where Q is a toric monoid. C has 3 kinds of points:

- General points, étale locally

$$\underline{C} \cong \mathrm{Spec} A[x].$$

  Chart for log structure is $Q \to A[x], q \mapsto \varphi(q)$.

- Marked points

$$\underline{C} \cong \mathrm{Spec} A[x].$$

  Log chart $Q \oplus \mathbb{N} \to A[x], (q, n) \mapsto \varphi(q)x^n$

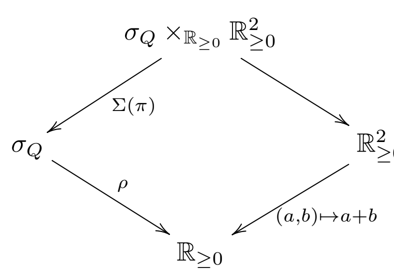
- Nodal points

$$\underline{C} \cong \mathrm{Spec} A[x, y]/(xy - \varphi(\rho)),$$

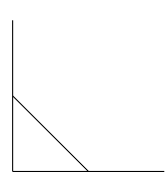
  for some $\rho \in Q, \rho \neq 0$ and chart $Q \oplus_{\mathbb{N}} \mathbb{N}^2 \to A[x, y]/(xy - \varphi(\rho)), (q, (a, b)) \mapsto \varphi(q)x^a y^b$, and $Q \oplus_{\mathbb{N}} \mathbb{N}^2$ is defined using maps $1 \to \rho, 1 \to (1, 1)$. $Q \oplus_{\mathbb{N}} \mathbb{N}^2 = Q \oplus \mathbb{N}^2 / \sim, (\alpha + \rho, (\beta, \gamma)) \sim (\alpha, (\beta + 1, \gamma + 1))$.

Tropicalize $C \to W = (\mathrm{Spec} k, k^* \oplus Q)$, where $\alpha(r, q) = \begin{cases} r, & q = 0 \\ 0, & q \neq 0 \end{cases}$. $\Sigma(\pi) : \Sigma(C) \to \Sigma(W) = \sigma_Q = \mathrm{Hom}(Q, \mathbb{R}_{\geq 0})$. Cones of $\Sigma(C)$ associated to 3 types of points:

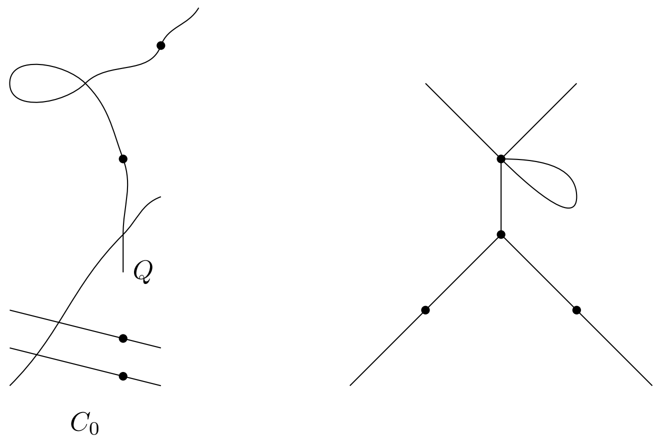
- Generic point $\eta \in C$ gives $\sigma_\eta = \sigma_Q$
- p a marked points of C. $\overline{\mathcal{M}}_{C,p} = Q \oplus \mathbb{N}$, $\Sigma(\pi) : \sigma_p = \sigma_Q \times \mathbb{R}_{\geq 0} \xrightarrow{\mathrm{pr}_1} \sigma_Q$.
- q a node. $\bar{\mathcal{M}}_{C,q} = Q \oplus_{\mathbb{N}} \mathbb{N}^2$ where the maps $\mathbb{N} \to Q, \mathbb{N} \to \mathbb{N}^2$ are given by $1 \mapsto \rho$ and $1 \mapsto (1, 1)$. $\sigma_q = \mathrm{Hom}(\overline{\mathcal{M}}_{C,q}, \mathbb{R}_{\geq 0}) = \sigma_Q \times_{\mathbb{R}_{\geq 0}} \mathbb{R}^2_{\geq 0}$.



  $\Sigma(\pi)^{-1}(m) = \{(a, b) \in \mathbb{R}^2_{\geq 0} | a + b = \rho(m)\}$, which is a line segment.



Example. *Fiber of* $\Sigma(\pi) : \Sigma(C) \to \Sigma(W)$ *over* $m \in \Sigma(W)$.



Stable log maps

Let $X \to S$ be a log morphism. A stable log map with target X/S is a diagram

$$\begin{array}{ccc} C & \xrightarrow{f} & X \\ {\scriptstyle\pi}\downarrow & & \downarrow \\ W & \longrightarrow & S \end{array}$$

and sections $p_1, ..., p_n : \underline{W} \to \underline{C}$ whose images coincide with the marked points of C(written as $(f : C/W \to X/S, p_1, ...p_n)$) where

- π is a family of log curves
- $\underline{f} : \underline{C}/\underline{W} \to \underline{X}$ is a stable map in ordinary sense.

Main Theorem.

Theorem. *(G-Siebert-Abramovich-Chen-Marcus-Wise)*

- $\exists$ *a moduli space* $\mathscr{M}_\beta(X/S)$ *of stable log maps of "type* β*"(*β*: curves class + genus + #marked points + contact orders)*
- *If* $X \to S$ *is proper,* $\mathscr{M}_\beta(X/S) \to S$ *is proper*
- *If further* $X \to S$ *is log smooth, then* $\mathscr{M}_\beta(X/S)$ *carries a virtual fundamental class*

We will talk tomorrow about this moduli space.

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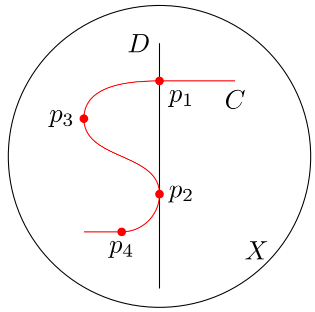
(X,D), $D \subseteq X$ smooth divisor in smooth X. $\exists (C, p_1, ..., p_n) \xrightarrow{f} (X, D)$ if $f^{-1}(D) \subseteq \{p_1, ..., p_n\}$.

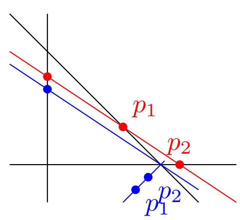


$$\begin{array}{rccc} f^\flat : & \mathcal{M}_{X,f(p_i)} & \to & \mathcal{M}_{C,p_i} \\ \bar{f}^\flat : & \overline{\mathcal{M}}_{X,f(p_i)} & \to & \overline{\mathcal{M}}_{C,p_i} \\ f(p_i) \in D : & \mathbb{N} & \xrightarrow{u_{p_i}} & \mathbb{N} \\ f(p_i) \notin D : & 0 & \xrightarrow{0} & \mathbb{N} \end{array}$$

$t = 0$ a local equation for D. v a local parameter for C at p_i. $t \mapsto f^*(t) = t \cdot f = \varphi v^{u_{p_i}}$, where φ is invertible. u_{p_i} is order of tangency of C at p_i with D.

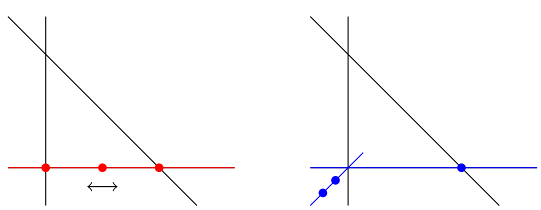
In general, given stable log map $f : (C, p_1, ..., p_n) \to X$, where C over $W = (\mathrm{Spec}k, k^* \oplus Q)$, $\bar{f}^\flat : \overline{\mathcal{M}}_{X,f(p_i)} \to \overline{\mathcal{M}}_{C,p_i} = Q \oplus \mathbb{N} \xrightarrow{\mathrm{pr}_2} \mathbb{N}$. This composition u_{p_i} is the contact order at p_i.

Example. $(X,D) = (\mathbb{P}^2, L_0 \cup L_1 \cup L_2)$. *Genus 0, degree 1.*

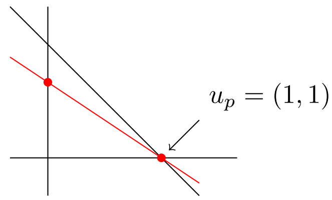


$\overline{\mathcal{M}}_{X,f(p_i)} = \mathbb{N}^2 \xrightarrow{u_{p_i}} \mathbb{N}$. $u_{p_1} = (1,0), u_{p_2} = (0,1)$.

The curve can degenerate entirely into one of the divisors. And the marked points can move around as well.



We can also pose different contact order conditions, e.g. $u_p = (1,1)$*. Then* p *cannot move around anymore.*



More on $\mathscr{M}_\beta(X/S)$ which is a log DM stack. β : genus, degree, #marked points, contact order. Does not classify all possible stable log maps.

$$\begin{array}{ccccccc} (C, \mathcal{M}_C) & \longrightarrow & (X, \mathcal{M}_X) & \rightsquigarrow & (C, \mathcal{M}_C \oplus \mathbb{N}^r) & \longrightarrow & (X, \mathcal{M}_X \oplus \mathbb{N}^r) \\ \downarrow & & & & \downarrow & & \\ (W, \mathcal{M}_W) & & & & (W, \mathcal{M}_W \oplus \mathbb{N}^r) & & \end{array}$$

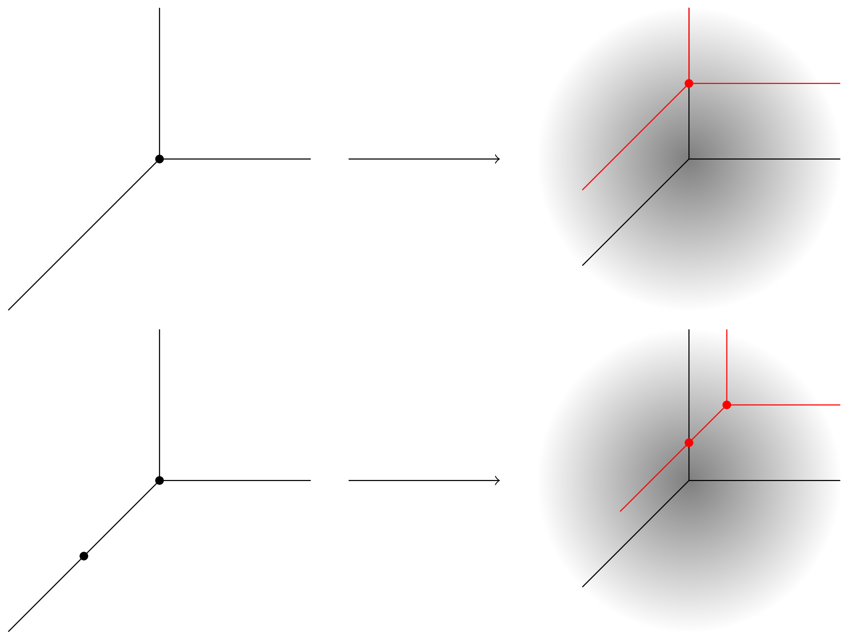
Main problem is that the category of diagram like this won't be finite and possibly have infinite automorphism. So we introduce basicness.

$\mathscr{M}_\beta(X/S)$ classifies <u>basic</u> stable log maps. Basicness is defined pointwise on W.

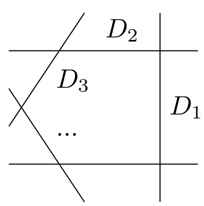
$$\begin{array}{ccccccc} C & \longrightarrow & X & \xrightarrow{\text{Tropicalize}} & \Sigma(C) & \xrightarrow{\Sigma(f)} & \Sigma(X) \\ \downarrow & & & & \downarrow & & \\ W = (\mathrm{Spec} k, k^* \oplus Q) & & & & \Sigma(W) = \sigma_Q & & \end{array}$$

The family is basic if this family of tropical curves is the moduli space of tropical curves in $\Sigma(X)$ of a fixed type.

Example. *For the vanilla case,* $Q = 0$*. For the case that the curve degenerates into one of the divisors,* $Q = \mathbb{N}$*. Further, for the case when one of the marked points moves to one of the intersection points of divisors,* $Q = \mathbb{N}^2$*.*

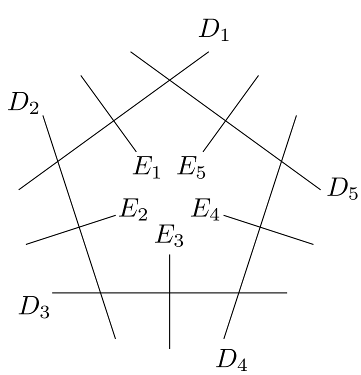


Now we start talking about mirror symmetry. Mirrors to log CY surfaces(G-Hacking-Keel). Fix (X, D) where X is a projective non-singular rational surface, $D \in |-K_X|$ which is a cycle of $\mathbb{P}^1$'s.



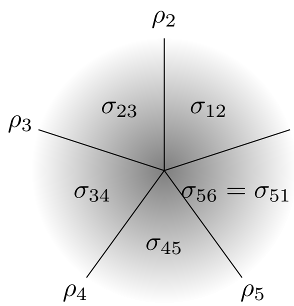
$$D = D_1 + ... + D_n$$

We will illustrate the process by an example. Let $\mathrm{dP}_5 =$ del Pezzo surface of degree 5. $-K_X$ can be represented by a cycle of 5 -1 curves.



$$E_i.D_j = \delta_{ij}$$

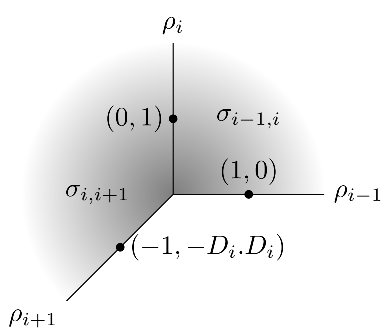
Consider $B = \Sigma(X, D)$:



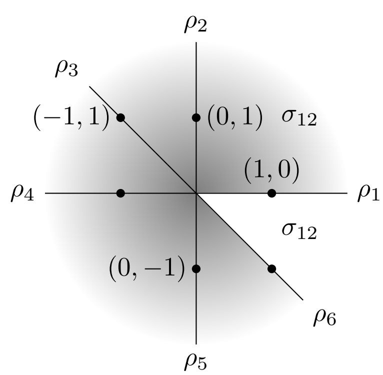
$\rho_i \leftrightarrow D_i$

$\sigma_{i,i+1} \leftrightarrow D_i \cap D_{i+1}$

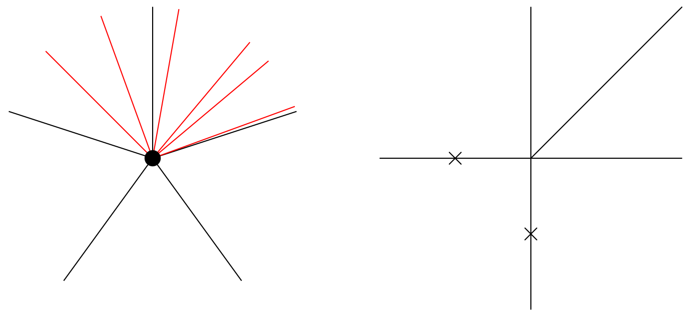
1. Give $B \backslash \{0\}$ an affine structure. Let $V_i = \text{Int}(\sigma_{i-1,i} \cap \sigma_{i,i+1})$. Identify $\overline{V}_i$ with the subset of $\mathbb{R}^2$.



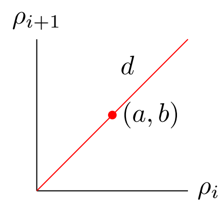
This comes from the intersection theory. For a fan picture of three rays given by $(0,1), (1,0)$ and $(-1, -D_i.D_i)$, the divisor corresponding to $(0,1)$ has a self-intersection number $D_i.D_i$. So for dP_5 we have:



Glue the two σ_{12}s. There has to be a singularity at the origin. So we have the following affine manifold with a singular point at the origin. Bernd would put the singularity on the edges and have another affine manifold.



2. Build the canonical scattering diagram on B. Fix $d \subseteq \sigma_{i,i+1}$ a ray of rational slope.



$\gcd(a,b) = 1$

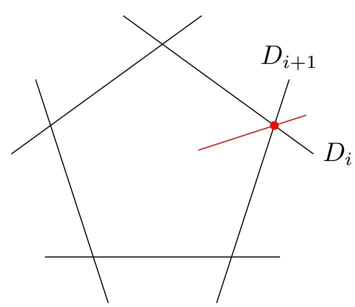
Attach a function f_d to d.

$$f_d = exp(\sum_{\beta} k_\beta N_\beta z^\beta (x_i^a x_{i+1}^b)^{-k_\beta})$$

To make sense of this, first fix a toric monoid $P \subseteq \mathrm{H}_2(X, \mathbb{Z})$ with $P^* = \{0\}$ and if $\beta \in \mathrm{H}_2(X, \mathbb{Z})$ represents an effective curve, then $\beta \in P$. We write $z^\beta \in k[P]$. x_i, x_{i+1} variables to be explained. We sum over all $\beta \in P$ with

$$\beta.D_j = \begin{cases} k_\beta a, & j = i \\ k_\beta b, & j = i+1 \\ 0, & j \notin \{i, i+1\} \end{cases}$$

$N_\beta = \#$ of genus 0, 1−pointed curves representing the class β with contact order $u_p = (k_\beta a, k_\beta b)$. We use virtual fundamental class to calculate N_β. Note $N_\beta \neq 0 \Rightarrow \beta \in P$.



We collect all these rays: $\mathscr{D} = \{(d, f_d)\}$. Now back to dP_5. The only nonzero f_ds are those associated to ρ_i whose nonzero terms are given by $\beta =$ multiple covers of E_i. And Pandharipande knows $N_{kE_i} = \frac{(-1)^{k+1}}{k^2}$. So

$$f_{\rho_i} = exp(\sum_k k \frac{(-1)^{k+1}}{k^2} z^{kE_i} x_i^{-k}) = 1 + z^{E_i} x_i^{-1}.$$

What do we do with a collection $\mathscr{D}$ of rays?

Pick a monomial ideal $I \subseteq k[P]$ such that $A_I := k[P]/I$ is Artinian. Goal: Build an affine scheme flat over $\mathrm{Spec} A_I$. For $i = 1, ..., n$ let

$$U_i = \mathrm{Spec} A_I[x_{i-1}, x_i^{\pm 1}, x_{i+1}]/(x_{i-1}x_{i+1} - z^{[D_i]} x_i^{-D_i^2} f_{\rho_i}),$$

where ρ_i is the ray of B corresponding to D_i and f_{ρ_i} is the function associated to direction $(1, 0)$. U_i canonically contains open subsets

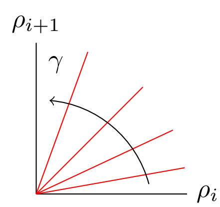
$$U_{i-1,i} = \{x_{i-1} \neq 0\} = \mathrm{Spec} A_I[x_{i-1}^{\pm 1}, x_i^{\pm 1}]$$

$$U_{i,i+1} = \{x_{i+1} \neq 0\} = \mathrm{Spec} A_I[x_{i+1}^{\pm 1}, x_i^{\pm 1}].$$

Both canonically isomorphic to $\mathrm{Spec} A_I \times (k^*)^2$. Now glue U_i to U_{i+1} by

$$U_i \supseteq U_{i,i+1} \xrightarrow[\theta_{\gamma,\mathscr{D}}]{\cong} U_{i,i+1} \subseteq U_{i+1},$$

where $\theta_{\gamma,\mathscr{D}}$ is a composition of wall-crossing automorphisms. When we cross $(\mathbb{R}_{\geq 0}(a, b), f_{(a,b)})$, we use $x_i \mapsto x_i f^b, x_{i+1} \mapsto x_{i+1} f^{-a}$ as the wall-crossing automorphism.



This gives a scheme $\mathcal{X}_I^\circ \to \mathrm{Spec} A_I$. $\mathcal{X}_I^\circ$ is an infinitesimal deformation of $\mathbb{V}_n \backslash \{0\}$, where $\mathbb{V}_n = \mathbb{A}^2_{x_1,x_2} \cup ... \cup \mathbb{A}^2_{x_n,x_1} \subseteq \mathbb{A}^n_{x_1,...,x_n}$. We haven't choose the scattering diagram yet. If we choose the canonical scattering diagram, then we will have the following diagram.

Theorem. *(GHK). $\mathcal{X}_I = \mathrm{Spec}\Gamma(\mathcal{X}_I^\circ, \mathcal{O}_{\mathcal{X}_I^\circ}) \to A_I$ is a flat deformation of $\mathbb{V}_n$.*

Tomorrow we will talk about the theorem further.

20170921 Mark Gross Cambridge(VC KIAS)

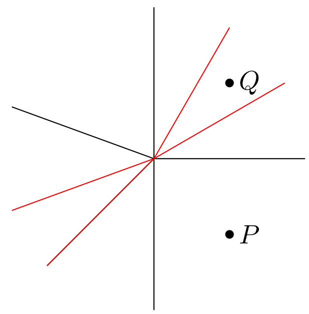
Back to the theorem.

Theorem. *(GHK). $\mathcal{X}_I = \mathrm{Spec}\Gamma(\mathcal{X}_I^\circ, \mathcal{O}_{\mathcal{X}_I^\circ}) \to A_I$ is a flat deformation of $\mathbb{V}_n$.*

Key point: need surjectivity

$$\Gamma(\mathcal{X}_I^\circ, \mathcal{O}_{\mathcal{X}_I^\circ}) \to \Gamma(\mathbb{V}_n, \mathcal{O}_{\mathbb{V}_n^\circ}) = \Gamma(\mathbb{V}_n, \mathcal{O}_{\mathbb{V}_n}).$$

$\Gamma(\mathbb{V}_n, \mathcal{O}_{\mathbb{V}_n^\circ})$ has basis $\{x_i^a x_{i+1}^b | 1 \leq i \leq n, a, b \in \mathbb{N}\} \leftrightarrow B(\mathbb{Z})$ which corresponding to $p \in B(\mathbb{Z})$. We build a theta function $\theta_p \in \Gamma(\mathcal{X}_I^\circ, \mathcal{O}_{\mathcal{X}_I^\circ})$.



$$\theta_{P,Q} = \sum_\gamma \text{Mono}(\gamma),$$

where Mono(γ) is the final monomial on γ.

For the example dP$_5$, $\theta_i = \theta_{P_i}$, where P_is are the primitive point for each ρ_i.

Exercise. $\theta_{i-1}\theta_{i+1} = z^{[D_i]} \cdot (\theta_i + z^{[E_i]})$.

Now we will see a simple product rule which will tell us the result in the exercise. Given $p, q \in B(\mathbb{Z})$,

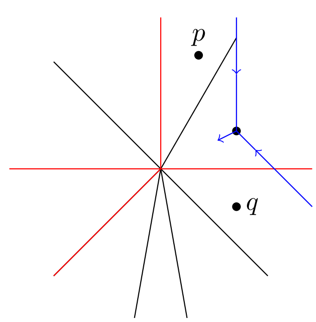
$$\theta_p \cdot \theta_q = \sum_{r \in B(\mathbb{Z})} \alpha_{pqr}\theta_r, \alpha_{pqr} \in A_I,$$

where $\alpha_{pqr} = \sum_{\gamma_p, \gamma_q} c_{\gamma_p} c_{\gamma_q}$ where we sum over all broken lines γ_p, γ_q with asymptotic directions p, q respectively, basepoint $r \in \sigma_{i,i+1} = \mathbb{R}^2_{\geq 0}, r = (a, b)$.

$$\text{Mono}(\gamma_p) = c_{\gamma_p} x_i^{a_p} x_{i+1}^{b_p}$$

$$\text{Mono}(\gamma_q) = c_{\gamma_q} x_i^{a_q} x_{i+1}^{b_q}$$

with $(a, b) = (a_p + a_q, b_p + b_q)$.



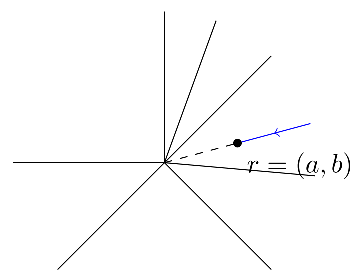
To prove the product rule, expand θ_p, θ_q at base-point r.

$$\theta_p = \sum_{\gamma_p} \text{Mono}(\gamma_p)$$

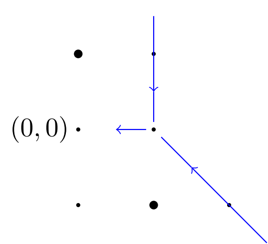
$$\theta_q = \sum_{\gamma_q} \text{Mono}(\gamma_q).$$

Look at coefficient of $x_i^a x_{i+1}^b$ in the expanded product $\sum_{\gamma_p,\gamma_q} \text{Mono}(\gamma_p)\text{Mono}(\gamma_q)$. The formula given for α_{pqr} is the coefficient of $x_i^a x_{i+1}^b$.

Claim. if $x_i^a x_{i+1}^b$ appears in some θ_s expanded at r, then $s = r$ and the monomial has coefficient 1. Prove by picture. Only one possible broken line has asymptotic direction r.



For example, when we take the product of two monomials in $k[x_1^{\pm 1}, x_2^{\pm 2}]$, there is a picture to explain this. The philosophy behind is the same.



<u>General Mirror Construction</u>(G-Siebert) Start with a log smooth pair (X, D) over Spec$k$, e.g. D is normal crossings. Will be able to define a ring in cases either $\pm(K_X + D)$ is nef or (X, D) is log CY, i.e. $X \backslash D$ carries a nowhere vanishing top dimensional holomorphic form with at most simple poles along D $\Rightarrow K_X + D = \sum a_i D_i, a_i \geq 0$. We will stick to the first case while the second will spare us the worry of the existence of the minimal model.

Let $B = \Sigma(X)$ (no affine structure). $B(\mathbb{Z})$ set of integral points in B. $P \subseteq \text{H}_2(X, \mathbb{Z})$ a toric monoid containing all classes of effective curves on X. $I \subseteq k[P]$ a monomial ideal with $A_I = k[P]/I$ Artinian. Goal: Define an A_I-algebra structure on the A_I-module

$$R_I = \oplus_{p \in B(\mathbb{Z})} A_I \cdot \theta_p$$

$$\theta_p \cdot \theta_q = \sum_{r\in B(\mathbb{Z})} \alpha_{pqr}\theta_r, \alpha_{pqr} \in A_I$$

$$\alpha_{pqr} = \sum_{\beta\in P\setminus I} N^{\beta}_{pqr} z^{\beta}, N^{\beta}_{pqr} \in \mathbb{Q} \subseteq k.$$

Define of the N^{β}_{pqr}: For $r \in B(\mathbb{Z})$ there is a minimal cone of $\Sigma(X)$ containing r, corresponding to a stratum Z_r of X. Pick a general $z \in Z_r$. $N^{\beta}_{pqr} = \#$ 3−pointed genus 0 curves of class β with contact orders p, q and $-r$ at the 3 points, and the point with contact order with $-r$ maps to the chosen point z.

Negative order of tangency → punctured invariants(joint with Abramovich-Chen-G-S). Suppose given $C \to W$ log smooth family of curves with section $p : W \to C$ disjoint from all special points on C. To mark the point p, we use the log structure $\mathcal{M}_C \oplus_{\mathcal{O}^*_C} \mathcal{M}_{(\underline{C},p)} \to \mathcal{O}_C, (s_1, s_2) \mapsto \alpha(s_1)\alpha(s_2), i.e. (\varphi s_1, s_2) = (s_1, \varphi s_2)$ for invertible φ. To puncture the point, consider a subsheaf

$$\mathcal{E} \subseteq \mathcal{M}_C \oplus_{\mathcal{O}^*_C} \mathcal{M}^{\text{gp}}_{(\underline{C},p)}$$

where gp indicates the Grothendieck group. Here

$$\mathcal{E} = \{(s_1, s_2)|s_2 \in \mathcal{M}_{(\underline{C},p)} \text{ if } \alpha(s_1) \neq 0\}.$$

$$\alpha : \mathcal{E} \to \mathcal{O}_C, \alpha(s_1, s_2) = \begin{cases} \alpha(s_1)\alpha(s_2), & \text{if } \alpha(s_1) \neq 0 \\ 0, & \text{if } \alpha(s_1) = 0 \end{cases}.$$

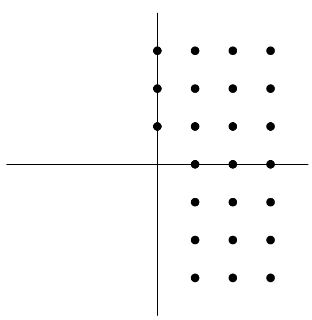
Example. $\mathcal{M}_C = \mathcal{O}^*_C \Rightarrow \mathcal{E} = \mathcal{M}_{(\underline{C},p)}$.

When C is smooth curve, $\mathcal{M}_C = \mathcal{O}^*_C \oplus \mathbb{N}$,

$$\alpha(s, n) = \begin{cases} s, & n = 0 \\ 0, & n \neq 0 \end{cases}.$$

$$\overline{\mathcal{E}} \subseteq \overline{\mathcal{M}}_{C,p} \oplus \overline{\mathcal{M}}^{\text{gp}}_{(\underline{C},p),p} = \mathbb{N} \oplus \mathbb{Z},$$

$$\overline{\mathcal{E}}_p = \{(a, b)|b \in \mathbb{N} \text{ if } a = 0\}.$$



Not finitely generated,
hence not fs

$\overline{\mathcal{E}}$ is not preserved under infinitesimal definitions. So deformation of stable maps

$$\begin{array}{ccc} C & \longrightarrow & X \\ \downarrow & & \\ W & & \end{array}$$

is locally obstructed. Need to work with a relative obstruction Theory over an Artin stack which encodes a combinatorial obstruction to deforming, not necessarily pure dimensional. So virtual fundamental class is also affected. In special cases, can prove equi-dimensional, including in definition of N^{β}_{pqr}. Contact order at punctures: $\overline{f}^{\flat} : \overline{\mathcal{M}}_{X,f(p)} \to \overline{\mathcal{E}}_p \subseteq \overline{\mathcal{M}}_{C,p} \oplus \mathbb{Z} \xrightarrow{\mathrm{pr}_2} \mathbb{Z}$. And the composition u_p is the contact order.

Example. *X is a smooth surface, D a -1 curve on X $\underline{f} : \underline{C} \to \underline{X}$ an isomorphism with D, then puncture $(C, p) \to X$ exists. $\mathcal{M}_C = \mathcal{O}^*_C \oplus \mathbb{N}$.*

Back to $X = \mathrm{dP}_5$ with divisors $D = \sum_i D_i$ and $E_i, i = 1, ..., 5$. So what is $\theta_{i-1}\theta_{i+1}$?

θ_0 is the unit of the ring. $r = 0, Z_r = X, z \in X$. Need curves C meeting D_{i-1}, D_{i+1} transversally at one point and passing through z. $[C].D_j = 0$ for $j \notin \{i-1, i+1\}$. So $C = D_i + E_i$. $\dim|D_i + E_i| = 1$. Thus $\exists$ a unique curve in this linear system passing through z. Coefficient of θ_0 is $z^{D_i+E_i}$. Coefficient of $\theta_i, Z_r = D_i, z \in D_i$. Curves transversal to D_{i-1}, D_{i+1} and order tangency with D_i being -1 at the point z: $C.D_{i-1} = 1, C.D_{i+1} = 1, C.D_i = -1$. So $C = D_i$. Coefficient is z^{D_i}. So

$$\theta_{i-1}\theta_{i+1} = z^{D_i+E_i}\theta_0 + z^{D_i}\theta_i.$$